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TRANSLATION

APPLICATION OF THEORY OF OPTIMAL PROCESSES TO FUNCTION APPROXIMATION PROBLEMS

BY

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TO FUNCTION APPROXIMATION PROBLEMS

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APPLICATION OF THEORY OF OPTIMAL PROCESSES TO FUNCTION APPROXIMATION PROBLEMS

V. G. Boltyanskiy

Let $F(x, y)$ be a function defined and continuous for all real values of the arguments. Then for any functions $x(t)$ and $y(t)$ given on interval $a \leq t \leq b$, the quantity

$$J = \int_a^b F(x(t), y(t))dt \tag{1}$$

can be used to compare the functions $x(t)$ and $y(t)$. For example, if $F(x,y) = (x - y)^2$ integral (1) assumes the form

$$J = \int_a^b (x(t) - y(t))^2 dt \tag{2}$$

and in this case represents the square of the distance between the elements $x(t)$ and $y(t)$ on space L_2. (Here and henceforth all functions of the argument t will be considered on one fixed interval $a \leq t \leq b$.)

The present article deals with the solution to the following problem. Given are functions $F(x, y)$ and $y(t)$. Also given is integer $n \geq 0$ and the real number $\alpha \geq 0$. Among all n times continously differentiable functions $x(t)$ given on the segment $a \leq t \leq b$ and having the property that the function $x^{(n)}(t)$ satisfies Lipshits'

condition with the constant α, we will find a function such that integral (1) assumes the minimum value. Henceforth, we shall call this problem the fundamental problem.

In the special case, when $F(x,y) = (x - y)^2$ (i.e., instead of functional (1) we consider (2)) and the number $\alpha = 0$, we arrive at the problem of finding that nth degree polynomial in $x(t)$ which has the least square deviation from the given function $y(t)$ over the interval $a \leq t \leq b$, i.e., the classical problem of finding the Fourier coefficients in the expansion of the function $y(t)$ by Legendre polynomials. Thus, the fundamental problem being considered here is a generalization of this classical problem.

First of all we shall show in section 1 that when certain natural requirements are imposed on the function $F(x,y)$, the fundamental problem just formulated always (i.e., for any function $y(t)$) has at least one solution, and that when functional (2) is being considered this problem has exactly one solution (for any function $y(t)$).

The problem of finding a function $x(t)$ which is a solution to the fundamental problem is considered in section 2. The maximum principle, which was discovered and proved in papers on the theory of optimal processes published by L. S. Pontryagin, R. V. Gamkrelidze, and the author of the present article (cf. [1]-[5]), is used to find the solution. The "problem of finding a highway profile" is also considered in section 2 as an example.

1. Existence of a solution. The question of the existence of a solution to the fundamental problem (and the question of the uniqueness of the solution in special case (2)) is considered in the following theorem.

Theorem 1. Let the function $F(x,y)$ be defined for all real

values of the arguments x and y and have the property that when y varies over any finite interval, the function $F(x, y)$ tends uniformly (with respect to y) toward $+\infty$, as $x \to +\infty$. Then the fundamental solution has at least one solution for any continuous function $y(t)$. If, in particular, $F(x, y) = (x - y)^2$, then this problem has exactly one solution for any continuous function $y(t)$.

Proof. The set of all real n times continuously differentiable functions $x(t)$ given for interval $a \leq t \leq b$ and having the property that their nth derivatives $x^{(n)}(t)$ satisfy Lipshits' condition with the constant α shall be denoted by $\Omega_\alpha^{(n)}$. Thus the inclusion $x \in \Omega_\alpha^{(n)}$ implies that the function $x(t)$ given for interval $[a, b]$ has n continuous derivatives in this interval and satisfies the inequality

$$|x^{(n)}(t') - x^{(n)}(t'')| \leq \alpha |t' - t''|$$

for any points t', t'' of interval $[a, b]$ The set $\Omega_a^{(n)}$ is obviously contained in the Banach space $C_{[a, b]}$ of all continuous functions given in interval $[a, b]$.

Fundamental Lemma. The set $\Omega_a^{(n)}$ is a closed, convex, locally compact subset of space $C_{[a, b]}$. Any closed bounded set contained in $\Omega_\alpha^{(n)}$ is compact.

This lemma is well known. For example, it follows easily from theorem 3.5.1 cited on page 127 of Timan's book [6]. In fact, let us use Σ_R to denote all functions $x \in \Omega_\alpha^{(n)}$ satisfying condition $\|x\| \leq R$, and $\Sigma_R^{(1)}$ to denote the set of all functions of the form $x^{(1)}(t)$, where $x \in \Sigma_R$. By virtue of the theorem cited above, the sets Σ_R, $\Sigma_R^{(1)}$, ..., $\Sigma_R^{(n)}$ are "compact in the space $C_{[a, b]}$ i.e., the closures of these sets in the space $C_{[a, b]}$ are compact. If x is an arbitrary accumulation point of the set Σ_R, then there exists a sequence x_1, x_2, ... of elements of the set Σ_R convergent on x. By converting, if necessary,

to the sequence we may consider (by virtue of the fact that closure of set $\Sigma_R^{(i)}$ is compact) that the sequence $x_1^{(i)}$, $x_2^{(i)}$, ... is convergent, $i = 1, 2, \ldots, n$. From this it follows by virtue of the theorem on the integration of uniformly convergent sequences, that the function $x(t)$ has continuous derivatives of orders $i = 1, 2, \ldots, n$, with $x^{(i)}$ being the limit of the sequence $x_1^{(i)}$, $x_2^{(i)}$, ... In particular, the function $x^{(n)}(t)$ as the limit of the sequence $x_1^{(n)}$, $x_2^{(n)}$, satisfies Lipshits' condition with the constant α, and therefore $x \in \Omega_\alpha^{(n)}$. Thus the set Σ_R is closed and consequently compact. The convexity of the set $\Omega_\alpha^{(n)}$ is obvious. Thus the fundamental theorem has been proved.

Let us return to a proof of Theorem 1. We shall denote by I the interval which contains the values of the function $y(t)$ when $t \in [a,b]$. Since if $y \in I$ the function $F(x,y)$ tends toward $+\infty$ uniformly with respect to $\underline{y}$, when $x \to \pm\infty$, the function $F(x,y)$ is bounded below for $y \in I$ and any $\underline{x}$. Thus there exists a non-negative number N such that

$$F(x, y) \geqslant -N \text{ for } y \in I \text{ and any } \underline{x}. \qquad (3)$$

Let us select arbitrary mutually different points a_0, a_1, ..., a_n which are inside interval $[a,b]$, and use $\varphi_i(t)$ to denote a polynomial of degree $\underline{n}$ which has value 1 at point a_i and value 0 at the remaining points a_j. Further we shall use ρ to denote a positive number small enough that in interval I_i of length ρ with center at point a_i $(i = 0, 1, \ldots, n)$ the polynomial $\varphi_i(t)$ takes on values greater than 2/3, while all other polynomials $\varphi_j(t)$ take on values less than $\frac{1}{3n}$ in absolute value. Finally, we use A to denote a positive number such that $|\varphi_i(t)| \leq A$ when $t \in [a,b]$, $i = 0, 1, \ldots, n$.

Now let $x(t)$ be an arbitrary function belonging to the set $\Omega_\alpha^{(n)}$, and let $\| x \| = \max\limits_{a \leq t \leq b} | x(t)|$ be its norm in the space $C_{[a,b]}$. We

use $\varphi(t)$ to denote a polynomial of degree n satisfying conditions $x^{(i)}(a) = \varphi^{(i)}(a)$, $i = 0, 1, \dots, n$. Then the function $x_1(t) = x(t) - \varphi(t)$ satisfies conditions $x_1^{(i)}(\alpha) = 0$, $i = 0, 1, \dots, n$. In addition, we have

$$x_1^{(n)}(t) = x_1^{(n)}(t) - x_1^{(n)}(a) = [x^{(n)}(t) - x^{(n)}(a)] - [\varphi^{(n)}(t) - \varphi^{(n)}(a)] = x^{(n)}(t) - x^{(n)}(a)$$

(since $\varphi^{(n)}(t)$ is a constant). Since function $x^{(n)}(t)$ satisfies Lipshits' condition with the constant α in interval $[a,b]$,

$$|x_1^{(n)}(t)| = |x^{(n)}(t) - x^{(n)}(a)| \leqslant \alpha(t-a) \leqslant \alpha(b-a) \quad \text{for} \quad t \in [a, b]. \tag{4}$$

Expanding the function $x_1(t)$ into a Taylor series, we find

$$x_1(t) = x_1(a) + \frac{t-a}{1!}x_1'(a) + \frac{(t-a)^2}{2!}x_1''(a) + \dots$$
$$\dots + \frac{(t-a)^{n-1}}{(n-1)!}x_1^{(n-1)}(a) + \frac{(t-a)^n}{n!}x_1^{(n)}(\theta),$$

where θ is an intermediate value between a and t. Since $x_1(a) = x_1'(a) = \dots = x_1^{(n-1)}(a) = 0$, when $t \in [a,b]$ we find by virtue of (4):

$$|x_1(t)| = \left|\frac{(t-a)^n}{n!}x_1^{(n)}(\theta)\right| \leqslant \frac{(t-a)^n}{n!}\cdot\alpha(b-a) \leqslant \alpha\cdot\frac{(b-a)^{n+1}}{n!}.$$

Thus $\|x_1\| \leqslant \alpha\cdot\frac{(b-a)^{n+1}}{n!}$ and therefore

$$\|\varphi\| = \|x - x_1\| \geqslant \|x\| - \|x_1\| \geqslant \|x\| - \alpha\cdot\frac{(b-a)^{n+1}}{n!}. \tag{5}$$

Let us now select a number $i = 0, 1, \dots, n$ such that the number $|\varphi(a_i)|$ is the largest among numbers $|\varphi(a_0)|$, $|\varphi(a_1)|$, $\dots$, $|\varphi(a_n)|$. By virtue of Lagrange's interpolation formula we have

$$\varphi(t) = \varphi(a_0)\varphi_0(t) + \varphi(a_1)\varphi_1(t) + \dots + \varphi(a_n)\varphi_n(t),$$

and therefore $|\varphi(t)| \leqslant (n+1)|\varphi(a_i)|A$ when $t \in [a, b]$, i.e.,

$$\|\varphi\| \leqslant (n+1)|\varphi(a_i)|A. \tag{6}$$

Combining inequalities (5) and (6), we obtain

$$|\varphi(a_i)| \geqslant \frac{\|\varphi\|}{(n+1)A} \geqslant \frac{1}{(n+1)A}\left[\|x\| - \alpha\cdot\frac{(b-a)^{n+1}}{n!}\right].$$

The inequalities

$$\varphi_i(t) > \frac{2}{3}, \quad \varphi_j(t) < \frac{1}{3n} \text{ when } j \neq i,$$

are fulfilled on interval I_i, and therefore we will have in this interval

$$\begin{aligned} |\varphi(t)| &= |\varphi(a_0)\varphi_0(t) + \varphi(a_1)\varphi_1(t) + \ldots + \varphi(a_n)\varphi_n(t)| \geqslant \\ &\geqslant |\varphi(a_i)\varphi_i(t)| - |\varphi(a_0)\varphi_0(t) + \ldots + \varphi(a_{i-1})\varphi_{i-1}(t) + \\ &+ \varphi(a_{i+1})\varphi_{i+1}(t) + \ldots + \varphi(a_n)\varphi_n(t)| \geqslant \\ &\geqslant |\varphi(a_i)| \cdot \{|\varphi_i(t)| - [|\varphi_0(t)| + \ldots + |\varphi_{i-1}(t)| + \\ &+ |\varphi_{i+1}(t)| + \ldots + |\varphi_n(t)|]\} > \\ &> \frac{1}{(n+1)A}\left[\|x\| - \alpha \cdot \frac{(b-a)^{n+1}}{n!}\right] \cdot \left(\frac{2}{3} - n \cdot \frac{1}{3n}\right) = \\ &= \frac{1}{3(n+1)A} \cdot \left[\|x\| - \alpha \cdot \frac{(b-a)^{n+1}}{n!}\right]. \end{aligned}$$

From this, finally, it follows that in interval I_i

$$\begin{aligned} |x(t)| &= |\varphi(t) + x_1(t)| \geqslant |\varphi(t)| - |x_1(t)| \geqslant \\ &\geqslant \frac{1}{3(n+1)A} \cdot \left[\|x\| - \alpha \cdot \frac{(b-a)^{n+1}}{n!}\right] - \alpha \cdot \frac{(b-a)^{n+1}}{n!}. \end{aligned} \tag{7}$$

Thus for any function $x \in \Omega_\alpha^{(n)}$ there is found a number $i = 0, 1, \ldots, n$ such that inequality (7) is fulfilled in interval I_i.

Now let J_0 be the value which functional (1) assumes for the function $x(t) = 0$. Further, let P be a positive number such that when $|x| > P$, $y \in I$ we will have

$$F(x, y) > \frac{J_0 + N(b-a)}{p}$$

(such a number P exists by virtue of the properties of the function $F(x,y)$ stated in the formulation of the theorem). Finally, let R be a positive number such that when $\|x\| > R$ the right side of relationship (7) is greater than P. Then for any function $x \in \Omega_\alpha^{(n)}$ satisfying the condition $\|x\| > R$ there will be found a number $i = 0, 1, \ldots, n$ such that inequality (7) is fulfilled in interval I_i and therefore the inequality $|x(t)| > P$ is fulfilled. It follows from this that

$$F(x(t), y(t)) > \frac{J_0 + N(b-a)}{\rho} \quad \text{when} \quad t \in I_1. \tag{8}$$

Also, by virtue of (3),

$$F(x(t), y(t)) \geq -N \quad \text{when} \quad t \in [a, b]. \tag{9}$$

Since the length of interval I_1 is equal to ρ, we get from inequalities (8) and (9)

$$\int_a^b F(x(t), y(t))\,dt > \frac{J_0 + N(b-a)}{\rho} \cdot \rho + (-N) \cdot [(b-a) - \rho] > J_0. \tag{10}$$

Thus, for any function $x \in \Omega_\alpha^{(n)}$ satisfying conditions $\| x \| > R$, inequality (10) is fulfilled.

Finally, let us use Σ_R to denote the set of all functions $x \in \Omega_\alpha^{(n)}$ satisfying condition $\|x\| \leq R$, and let J^* denote the lower bound of the values of functional (1) for the functions $x \in \Sigma_R$. Obviously $J_0 \geq J^*$ and therefore $\int_a^b F(x(t), y(t))\,dt \geq J^*$ for any function $x \in \Omega_\alpha^{(n)}$: this follows from inequality (10) when $\|x\| \geq R$ and from the definition of a lower bound when $\| x \| \leq R$. Therefore, to complete the proof of the first part of the theorem we need only establish that there exists such a function $x \in \Omega_\alpha^{(n)}$ for which functional (1) assumes the value J^*. This follows easily from the compactness of the set Σ_R (cf. fundamental lemma) and the continuity of integral (1) considered as a function of $x \in \Omega_\alpha^{(n)}$.

Thus, the first part of the theorem (existence of a solution) has been proven. Since in particular, the function $F(x,y) = (x - y)^2$ satisfies the conditions stated in Theorem 1, the formulated problem always has a solution also for functional (2). Let us show that in this case the solution is unique. Since functional (2) equals d^2, where $d = d(x, y)$ — the distance between the functionals $\underline{x}$ and $\underline{y}$ in the sense of the space metric L_2, and since the quantities $\underline{d}$ and d^2 reach their minima simultaneously, the problem reduces itself to

finding the element $x \in \Omega_\alpha^{(n)}$, for which $d(x,y) = \min$, i.e., to finding the point $x \in \Omega_\alpha^{(n)}$ closest to y. The space $C_{[a,b]}$ is naturally contained in L_2, with the straight lines in $C_{[a,b]}$ also being straight in L_2. Therefore the set $\Omega_\alpha^{(n)}$, convex in $C_{[a,b]}$ (cf. fundamental lemma) is also a convex subset of space L_2. But in space L_2 (by virtue of the strict convexity of its unit sphere) a convex subset may not contain more than one point closest to y*. Therefore in this case our fundamental problem has only one solution.

Thus Theorem 1 has been proved.

2. The Use of the Theory of Optimal Processes in Finding a Solution. Let $x(t)$ be an arbitrary function of the class $\Omega_\alpha^{(n)}$. Then the function $x^{(n)}(t)$ exists in interval $[a,b]$ and satisfies Lipshits' condition with the constant α and, consequently, is absolutely continuous. Therefore there exists almost everywhere a measurable function $u = x^{(n+1)}(t)$, with the relationship $|u(t)| \leq \alpha$ fulfilled at all points where the function $u(t)$ is defined. Thus, denoting the functions $x(t)$, $x^1(t)$, ..., $x^{(n)}(t)$, by x^1, x^2, ..., x^{n+1} respectively, we find that the following relationships are fulfilled:

$$\begin{cases} \dot{x}^1 = x^2, \\ \dot{x}^2 = x^3, \\ \dots\dots \\ \dot{x}^n = x^{n+1}, \\ \dot{x}^{n+1} = u(t). \end{cases} \tag{11}$$

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* We note that this property of convex sets (having only one closest point) is in a number of cases the characteristic property of convex sets. This has been proven by Buseman for the case of finite-dimensional Euclidian spaces. The more general cases (in particular, the case of compact subsets of space L_2) have been investigated very recently by N. V. Yefimov and S. B. Stechkin [7], [8], [9].

where $|u(t)| \leq \alpha$. These relationships are fulfilled almost everywhere in interval $[a, b]$ (everywhere even for the first n relationships). It is not difficult to see, what is the opposite of this, that if the absolutely continuous functions $x^1, x^2, \ldots, x^{n+1}$ almost everywhere in interval $[a, b]$ satisfy relationships (11), where $u(t)$ is some measurable function satisfying the condition $|u(t)| \leq \alpha$, then the function $x(t) = x^1(t)$ belongs in the class $\Omega_\alpha^{(n)}$. In fact, since the function $x^{i+1} (i = 1, 2, \ldots, n)$ is absolutely continuous and therefore continuous, it follows from relationship $\dot{x}^i = x^{i+1}$ that the absolutely continuous function x^i is the Riemann integral of continuous function x^{i+1}. Therefore function x^i has a continuous derivative equal to $x^{i+1} (i = 1, 2, \ldots, n)$ everywhere in interval $[a, b]$. Thus in interval $[a, b]$ the function $x^1(t)$ has a continuous nth derivative equal to $x^{n+1}(t)$, and this derivative, by virtue of relationships $\dot{x}^{n+1} = u(t)$, $|u(t)| \leq \alpha$, which hold nearly everywhere satisfies Lipshits' condition with the constant α, i.e., $x^1 \in \Omega_\alpha^{(n)}$.

Thus, instead of functions of the class $\Omega_\alpha^{(n)}$ we may consider (absolutely continuous) solutions to system (11) with the restriction $|u(t)| \leq \alpha$. Thus the fundamental problem is equivalent to the following optimum problem: in the class of measurable controls $u(t)$ satisfying the restriction $|u(t)| \leq \alpha$, we will find a control for which the solution to system (11) realizes a minimum for the integral

$$J = \int_a^b F(x^1, y(t))\,dt;$$

the end values $x^i(a)$ and $x^i(b)$, $i = 1, 2, \ldots, n + 1$ are arbitrary.

Since the integrand in the integral for J depends explicitly on t (by means of the given function $y(t)$), we shall introduce the auxiliary variable $x^{n+2} \equiv t$, which of course satisfies the differential

equation

$$\dot{x}^{n+2} = 1$$

with initial condition $x^{n+2}(a) = a$. Then the optimum problem being considered will assume the following form:

On space X of variables $x^1, x^2, \ldots, x^{n+1}, x^{n+2}$ there is given an initial manifold M_0 with equation $x^{n+2} = a$ and a finite manifold M_1 with equation $x^{n+2} = b$ (each of the manifolds has dimensionality n + 1). In the class of measurable controls u(t), satisfying the restriction $|u(t)| \leq \alpha$ find the control for which the solution to the system

$$\begin{cases} \dot{x}^1 = x^2, \\ \dot{x}^2 = x^3, \\ \cdots\cdots \\ \dot{x}^n = x^{n+1}, \\ \dot{x}^{n+1} = u, \\ \dot{x}^{n+2} = 1, \end{cases} \tag{12}$$

starting at the moment $t_0 = a$ from some point of the manifold M_0 and arriving (obviously, at the moment $t_1 = b$ — by virtue of Eq. (12) at the manifold M_1 brings about a minimum for the integral

$$\int_{t_0}^{t_1} F(x^1, y(x^{n+2}))\, dt.$$

This problem (equivalent to our fundamental problem) shall also be solved. For this purpose we shall make use of Theorems 1 and 3 from the literature [5]. The functions F(x,y) and y(t) will be assumed continuously differentiable. The use of the maximum principle requires the setting up of a function **H**, which for the optimal problem under consideration has the form

$$H = \psi_0 F(x^1, y(x^{n+2})) + \psi_1 x^2 + \psi_2 x^3 + \ldots + \psi_n x^{n+1} + \psi_{n+1} u + \psi_{n+2}. \tag{13}$$

With the aid of this function **H** we shall set up a system of differential equations for the auxiliary unknowns ψ_i:

$$\begin{cases} \dot\psi_0 = 0, \\ \dot\psi_1 = -\dfrac{\partial H}{\partial x^1} = -\psi_0 \dfrac{\partial F(x^1, u(x^{n+2}))}{\partial x^1}, \\ \dot\psi_2 = -\dfrac{\partial H}{\partial x^2} = -\psi_1, \\ \dot\psi_3 = -\dfrac{\partial H}{\partial x^3} = -\psi_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \dot\psi_{n+1} = -\dfrac{\partial H}{\partial x^{n+1}} = -\psi_n \end{cases} \qquad (14)$$

(we will not write out the expression for $\dot\psi_{n+2}$, since we do not need it).

Let $x(t)$ be a solution to the fundamental problem. Then, in accordance with what was said above, the functions

$$x^1 = x(t), \quad x^2 = x'(t), \ldots, \quad x^{n+1} = x^{(n)}(t), \quad x^{n+2} = t$$

yield a solution to the optimum problem considered above (cf. (12)). Therefore there exists a non-zero solution $\psi_0, \psi_1, \ldots, \psi_{n+1}, \psi_{n+2}$ to system (14) (completed by the equation for ψ_{n+2} which has not been written out) which satisfies the conditions stated in Theorem 1 in an article from the literature [5]. The maximum condition of the function H gives (almost everywhere in interval $[a,b]$):

$$\max_{-\alpha \le u \le \alpha} \psi_{n+1}(t) u = \psi_{n+1}(t) u(t),$$

i.e.,

$$u(t) \begin{cases} = \alpha \operatorname{sign} \psi_{n+1}(t), & \text{if } \psi_{n+1}(t) \neq 0, \\ \text{not defined}, & \text{if } \psi_{n+1}(t) = 0. \end{cases} \qquad (15)$$

Let us now write out the transversality conditions (Theorem 3 from the literature [5]). Since the vectors pointing along the axes $x^1, x^2, \ldots, x^{n+1}$ are parallel to the hyperplanes of M_0 and M_1, then the conditions of transversality have the form

$$\psi_i(a) = 0, \quad i = 1, 2, \ldots, n+1, \qquad (16)$$

$$\psi_i(b) = 0, \quad i = 1, 2, \ldots, n+1. \qquad (17)$$

By virtue of the first of Eqs. (14) we have ψ_0 = const, where, according to Theorem 1 in the literature [5], $\psi_0 \leq 0$. It is not difficult to see that the assumption $\psi_0 = 0$ leads to a contradiction. In fact, if $\psi_0 = 0$, then $\dot{\psi}_1 = 0$ (cf. (14)) and by virtue of (16), $\psi_1 \equiv 0$. From this we get $\dot{\psi}_2 = 0$ (cf. (14)) and, by virtue of (16), $\psi_2 \equiv 0$; etc. In this way we find successively $\psi_0 = \psi_1 = \psi_2 = \ldots = \psi_{n+1} \equiv 0$. Since function H is identically equal to zero along the optimal trajectory (Theorem 1 from the literature [5]), we get by virtue of (13), $\psi_{n+2} \equiv 0$. But this contradicts the fact that ψ_0, ψ_1, ψ_2, ..., ψ_{n+2} is a non-zero solution. Thus $\psi_0 < 0$ and we may assume that $\psi_0 = -1$ (since all the quantities ψ_i are defined with an accuracy to a common, constant, positive proportionality factor). System (14) now assumes the form (after the substitution $x^{n+2} = t$)

$$\begin{cases} \dot{\psi}_1 = \dfrac{\partial F(x^1, y(t))}{\partial x^1}, \\ \dot{\psi}_2 = -\psi_1, \\ \dot{\psi}_3 = -\psi_2, \\ \ldots\ldots\ldots\ldots \\ \dot{\psi}_{n+1} = -\psi_n, \end{cases}$$

from here, by taking into account the transversality conditions (16), we obtain

$$\psi_1(t) = \int_a^t \frac{\partial F(x^1(t), y(t))}{\partial x^1}\, dt,$$

$$\psi_2(t) = -\int_a^t \left(\int_a^t \frac{\partial F(x^1(t), y(t))}{\partial x^1}\, dt \right) dt,$$

$$\psi_3(t) = \int_a^t \left[\int_a^t \left(\int_a^t \frac{\partial F(x^1(t), y(t))}{\partial x^1}\, dt \right) dt \right] dt,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

and in general

$$\psi_k(t) = (-1)^{k-1} \int_a^t \int_a^t \ldots \int_a^t \frac{\partial F(x^1(t), y(t))}{\partial x^1}\, dt \ldots dt\, dt \qquad \text{(k quadrature)}$$

$$k = 1, 2, \ldots, n+1$$

According to the well known analysis formula

$$\underbrace{\int_a^t \int_a^t \cdots \int_a^t}_{k \text{ quadrature}} f(t)\,dt \ldots dt\,dt = \frac{1}{(k-1)!}\int_a^t (t-\xi)^{k-1} f(\xi)\,d\xi.$$

we can rewrite the value found for $\psi_k(t)$ in the form

$$\psi_k(t) = \frac{1}{(k-1)!}\int_a^t (\xi - t)^{k-1} \frac{\partial F(x^1(\xi), y(\xi))}{\partial x^1}\,d\xi, \quad k = 1, 2, \ldots, n+1. \tag{18}$$

The transversality conditions (17) now takes the form

$$\psi_k(b) = \frac{1}{(k-1)!}\int_a^b (\xi - b)^{k-1} \frac{\partial F(x^1(\xi), y(\xi))}{\partial x^1}\,d\xi = 0, \quad k = 1, 2, \ldots, n+1.$$

Multiplying this relationship by $(k-1)!c_{k-1}$ and summing with respect to $k = 1, 2, \ldots, n + 1$ we obtain

$$\int_a^b [c_0 + c_1(\xi - b) + c_2(\xi - b)^2 + \ldots + c_n(\xi - b)^n] \frac{\partial F(x^1(\xi), y(\xi))}{\partial x^1}\,d\xi = 0$$

for any values of the constants $c_0, c_1, \ldots, c_n$. Since $c_0 + c_1(\xi-b) + \ldots + c_n(\xi-b)^n$ is an arbitrary polynomial of degree $\leq n$, we can combine all transversality conditions (17) into one requirement

$$\int_a^b P(\xi) \frac{\partial F(x^1(\xi), y(\xi))}{\partial x^1}\,d\xi = 0$$

for any polynomial $P(\xi)$ of degree $\leq n$.

Further, formula (15) is rewritten, by virtue of (18), in the form:

$$u(t) \begin{cases} = \alpha \operatorname{sign}\left(\int_a^t (\xi - t)^n \frac{\partial F(x^1(\xi), y(\xi))}{\partial x^1}\,d\xi\right) & \text{if the expression in the parentheses is different from zero;} \\ \text{not defined, if this expression is equal to zero.} & \end{cases}$$

In other words, almost everywhere on the segment $[a,b]$ one of the relationships:

$$\int_a^t (\xi - t)^n \frac{\partial F(x^1(\xi), y(\xi))}{\partial x^1}\,d\xi = 0,$$

$$u(t) = \alpha \operatorname{sign}\left(\int_a^t (\xi - t)^n \frac{\partial F(x^1(\xi), y(\xi))}{\partial x^1}\,d\xi\right).$$

is fulfilled. Finally, from relationships (12) we obtain

$$u(t) = \frac{d^{n+1}}{dt^{n+1}} x^1(t)$$

(almost everywhere on interval [a,b]). Combining all that has been said and again denoting $x^1(t)$ by $x(t)$, we obtain the following statement.

Theorem 2. Let the functions $F(x,y)$ and $y(t)$ have continuous first derivatives. In order that the function $x(t)$ be a solution to the fundamental problem, it is necessary that one of the relationships:

$$\int_a^t (\xi - t)^n \frac{\partial F(x(\xi), y(\xi))}{\partial x} d\xi = 0,$$

$$x^{n+1}(t) = \alpha \operatorname{sign}\left(\int_a^t (\xi - t)^n \frac{\partial F(x(\xi), y(\xi))}{\partial x} d\xi\right)$$

be fulfilled almost everywhere on interval [a,b] and, in addition, that for any polynomial $P(t)$ of degree $\leq n$ the condition

$$\int_a^b P(t) \frac{\partial F(x(t), y(t))}{\partial x} dt = 0.$$

is fulfilled.

Example (problem of finding a highway profile). Considering functional (2), we arrive at the following problem when $n = 0$. Given are the function $y(t)$ differentiable on interval [a,b] and the number $\alpha \geq 0$. Find the function $x(t)$ satisfying Lipshits' condition with the constant α, for which integral (2) assumes the least possible value. This problem may be interpreted in the following way. It is required to build a highway between the two points A and B, where the contour of the locality between these points is given (the function $y(t)$), and in accordance with the conditions of highway operation the inclination of the road at any point should be $\leq \alpha$, i.e., the longitudinal section of the highway (its profile) should be described by

a function which satisfies Lipshits' condition with the constant α. To achieve this goal it is possible either to lay the road over the existing terrain, or to construct a causeway, or to cut through a trench for the road. If x(t) is the planned profile of the road, then let the cost of excavation (constructing causeways for sections where x(t) is greater than y(t) and excavating trenches for sections where x(t) is less than y(t)) be evaluated by integral (2). Find the most suitable (from the standpoint of material expenditures) profile for the highway.

The solution to this problem exists and is unique (Theorem 1). In order that the function x(t) be the desired solution it is necessary (by virtue of Theorem 2) that almost everywhere on interval [a,b], one of the relationships

$$\int_a^t [x(\xi) - y(\xi)]\, d\xi = 0, \tag{19}$$

$$x'(t) = \alpha \operatorname{sign}\left(\int_a^t [x(\xi) - y(\xi)]\, d\xi\right) \tag{20}$$

be fulfilled and, in addition, that the condition

$$\int_a^b [x(t) - y(t)]\, dt = 0. \tag{21}$$

be fulfilled.

If relationship (19) is fulfilled on some interval contained within [a,b], x(t) = y(t) on this interval, i.e., the road should be laid directly on the existing terrain. But if relationship (20) is fulfilled on some interval, then at the points of this interval x'(t) = $\pm\alpha$, so that on this interval the road consists of one or several pieces with inclination α or $-\alpha$. Thus the over-all characteristic of the road is that it consists of individual pieces laid on the existing terrain, and pieces of maximum allowable inclination,

which go along causeways or through trenches.

For example, let points A and B be located right at the place where a hollow (let us say a gully which the road must cross) occurs in the road between points A and B, so that a cross section of the terrain along the line AB has the shape shown in Fig. 1. We shall consider the line shown in Fig. 1 to be a graph of the function y(t), assuming that the abscissa coincides with straight line AB, and the abscissas of points A and B are equal respectively to a and b.

Fig. 1

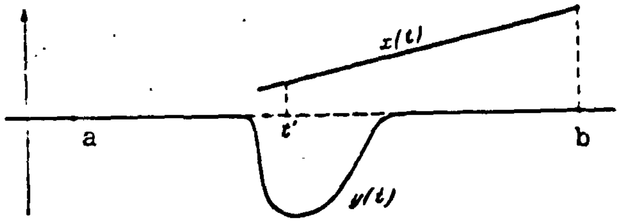

Fig. 2.

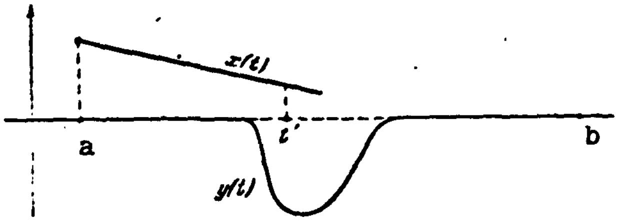

Fig. 3.

Let us assume that the walls of the gully are steep (having an inclination greater than α). The function, whose graph is the desired highway profile, will be denoted, as above, by x(t). It is easy to

see that $x(t) \leq 0$ for all $\underline{t}$. In fact if $x(t') > 0$, then, by virtue of the continuity of function $x(t)$, the inequality $x(t) > 0$ will also apply in some neighborhood of the point t'. Therefore, by slightly displacing point t', if necessary, we can ensure fulfillment of inequalities $x(t') > 0, \int_a^{t'}(x(t)-y(t))dt \neq 0.$ If inequality $\int_a^{t'}(x(t)-y(t))dt > 0,$ applies, then, by virtue of (20), $x'(t) = +\alpha$ when $t \geq t'$ (Fig. 2), and therefore

$$\int_a^{b}(x(t)-y(t))dt = \int_a^{t'}(x(t)-y(t))dt + \int_{t'}^{b}(x(t)-y(t))dt > 0,$$

which contradicts relationship (21). If inequality $\int_a^{t'}(x(t)-y(t))dt < 0,$ applies then by virtue of (20), $x'(t) = -\alpha$ when $t \leq t'$ (Fig. 3), but this implies that $x(t) > y(t)$ when $a \leq t \leq t'$, in spite of inequality $\int_a^{t'}(x(t)-y(t))dt < 0.$ The obtained contradiction indicates that $x(t) \leq 0$ for all $\underline{t}$.

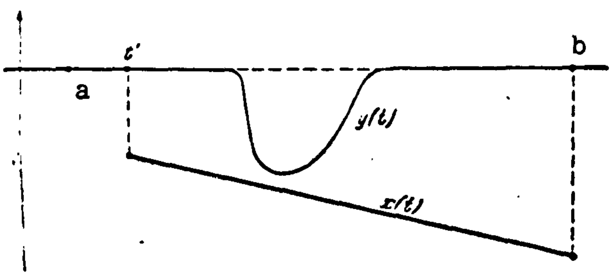

Fig. 4.

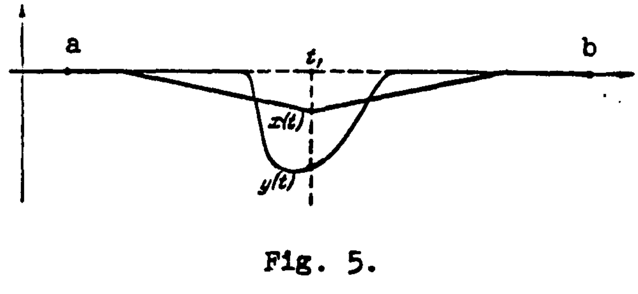

Fig. 5.

Let us now assume that inequality $x(t') < y(t')$ is fulfilled at some point t'. Moreover, if $\int_a^{t'}(x(t)-y(t))dt < 0$, then when $t > t'$ the highway descends with an inclination α, i.e., $x'(t) = -\alpha$ until the inequality $\int_a^{t}(x(t)-y(t))dt < 0$ breaks down. Furthermore the graphs of the functions $x(t)$ and $y(t)$ must necessarily intersect when $t > t'$, otherwise we would have $x(t) < y(t)$ for all $t > t'$ (Fig. 4), and therefore

$$\int_a^b(x(t)-y(t))dt = \int_a^{t'}(x(t)-y(t))dt + \int_{t'}^b(x(t)-y(t))dt < 0$$

in spite of relationship (21). Analogously, if inequalities $x(t') < y(t')$ and $\int_a^{t'}(x(t)-y(t))dt > 0$, are fulfilled, then when $t < t'$ the highway approaches point t' while ascending at inclination α (i.e., $x'(t) = +\alpha$), wherein when $t < t'$ the point of intersection between the graphs of $x(t)$ and $y(t)$ must occur.

Fig. 6.

All that has been said permits us to find a function $x(t)$ for different graphs of $y(t)$. Examples are given in Fig. 5 and 6. For the determination of point t_1 and the value $x(t_1)$ we have the relationship

$$\int_a^{t_1}(x(t)-y(t))dt = 0, \quad \int_{t_1}^{b}(x(t)-y(t))dt = 0,$$

while for the determination of points t_1 and t_2 in Fig. 6 we have the relationship

$$\int_a^{t_1}(x(t)-y(t))dt = 0, \quad \int_{t_2}^{b}(x(t)-y(t))dt = 0.$$

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